SLIM UNICORNS AND UNIFORM HYPERBOLICITY FOR ARC GRAPHS AND CURVE GRAPHS

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Abstract. We describe unicorn paths in the arc graph and show that they form 1–slim triangles and are invariant under taking subpaths. We deduce that all arc graphs are 7–hyperbolic. Considering the same paths in the arc and curve graph, this also shows that all curve graphs are 17–hyperbolic, including closed surfaces.

1. Introduction

The *curve graph* $\mathcal{C}(S)$ of a compact oriented surface S is the graph whose vertex set is the set of homotopy classes of essential simple closed curves and whose edges correspond to disjoint curves. This graph has turned out to be a fruitful tool in the study of both mapping class groups of surfaces and of hyperbolic 3–manifolds. One prominent feature is that $\mathcal{C}(S)$ is a *Gromov hyperbolic* space (when one endows each edge with length 1) as was proven by Masur and Minsky [MM99]. The main result of this paper is to give a new (short and self-contained) proof with a low uniform constant:

Theorem 1.1. *If $\mathcal{C}(S)$ is connected, then it is 17–hyperbolic.*

Here, we say that a connected graph Γ is *k–hyperbolic*, if all of its triangles formed by geodesic edge-paths are k–centred. A triangle is *k–centred at a vertex $c \in \Gamma^{(0)}$*, if c is at distance $\leq k$ from each of its three sides. This notion of hyperbolicity is equivalent (up to a linear change in the constant) to the usual slim-triangle condition [ABC+91].

After Masur and Minsky's original proof, several other proofs for the hyperbolicity of $\mathcal{C}(S)$ were given. Bowditch proved that k can be chosen to grow logarithmically with the complexity of S [Bow06]. A different proof of hyperbolicity was given by Hamenstädt [Ham07]. Recently, Aougab [Aou12], Bowditch [Bow12], and Clay, Rafi and Schleimer [CRS13] have proved, independently, that k can be chosen independent of S.

Our proof of Theorem 1.1 is based on a careful study of Hatcher's surgery paths in the arc graph $\mathcal{A}(S)$ [Hat91]. The key point is that these paths form 1–slim triangles (Section 3), which follows from viewing surgered arcs as *unicorn arcs* introduced as one-corner arcs in [HOP12]. We then use a hyperbolicity argument of Hamenstädt [Ham07], which provides a better constant than a similar criterion due to Bowditch [Bow12, Prop 3.1]. This gives rise to uniform hyperbolicity of the arc graph (Section 4) and then also of the curve graph (Section 5). Thus, we also prove:

Theorem 1.2. *$\mathcal{A}(S)$ is 7–hyperbolic.*

The arc graph was proven to be hyperbolic by Masur and Schleimer [MS13], and recently another proof has been given by Hilion and Horbez [HH12]. Uniform hyperbolicity, however, was not known.

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2. Preliminaries

Let S be a compact oriented topological surface. We consider arcs on S that are properly embedded and *essential*, i.e. not homotopic into ∂S. We also consider embedded closed curves on S that are not homotopic to a point or into ∂S. The *arc and curve graph* $\mathcal{AC}(S)$ is the graph whose vertex set $\mathcal{AC}^{(0)}(S)$ is the set of homotopy classes of arcs and curves on $(S, \partial S)$. Two vertices are connected by an edge in $\mathcal{AC}(S)$ if the corresponding arcs or curves can be realised disjointly. The *arc graph* $\mathcal{A}(S)$ is the subgraph of $\mathcal{AC}(S)$ induced on the vertices that are homotopy classes of arcs. Similarly, the *curve graph* $\mathcal{C}(S)$ is the subgraph of $\mathcal{AC}(S)$ induced on the vertices that are homotopy classes of curves.

Let a and b be two arcs on S. We say that a and b are in *minimal position* if the number of intersections between a and b is minimal in the homotopy classes of a and b. It is well known that this is equivalent to a and b being transverse and having no discs in $S - (a \cup b)$ bounded by a subarc of a and a subarc of b (*bigons*) or bounded by a subarc of a, a subarc of b and a subarc of ∂S (*half-bigons*).

3. Unicorn paths

We now describe Hatcher's surgery paths [Hat91] in the guise of unicorn paths.

Definition 3.1. Let a and b be in minimal position. Choose endpoints α of a and β of b. Let $a' \subset a, b' \subset b$ be subarcs with endpoints α, β and a common endpoint π in $a \cap b$. Assume that $a' \cup b'$ is an embedded arc. Since a, b were in minimal position, the arc $a' \cup b'$ is essential. We say that $a' \cup b'$ is a *unicorn arc obtained from* a^α, b^β. Note that it is uniquely determined by π, although not all $\pi \in a \cap b$ determine unicorn arcs, since the components of $a - \pi, b - \pi$ containing α, β might intersect outside π.

We linearly order unicorn arcs so that $a' \cup b' \leq a'' \cup b''$ if and only if $a'' \subset a'$ and $b' \subset b''$. Denote by $(c_1, \ldots, c_{n-1})$ the ordered set of unicorn arcs. The sequence $\mathcal{P}(a^\alpha, b^\beta) = (a = c_0, c_1, \ldots, c_{n-1}, c_n = b)$ is called the *unicorn path between* a^α *and* b^β.

The homotopy classes of c_i do not depend on the choice of representatives of the homotopy classes of a and b.

Remark 3.2. Consecutive arcs of the unicorn path represent adjacent vertices in the arc graph. Indeed, suppose $c_i = a' \cup b'$ with $2 \leq i \leq n-1$ and let π' be the first point on $a - a'$ after π that lies on b'. Then π' determines a unicorn arc. By definition of π', this arc is c_{i-1}. Moreover, it can be homotoped off c_i, as desired. The fact that c_0c_1 and $c_{n-1}c_n$ form edges follows similarly.

We now show the key 1–slim triangle lemma.

Lemma 3.3. *Suppose that we have arcs with endpoints $a^\alpha, b^\beta, d^\delta$, mutually in minimal position. Then for every $c \in \mathcal{P}(a^\alpha, b^\beta)$, there is $c^* \in \mathcal{P}(a^\alpha, d^\delta) \cup \mathcal{P}(d^\delta, b^\beta)$, such that c, c^* represent adjacent vertices in $\mathcal{A}(S)$.*

Proof. If $c = a' \cup b'$ is disjoint from d, then there is nothing to prove. Otherwise, let $d' \subset d$ be the maximal subarc with endpoint δ and with interior disjoint from c. Let $\sigma \in c$ be the other endpoint of d'. One of the two subarcs into which σ divides c is contained in a' or b'. Without loss of generality, assume that it is contained in a', denote it by a''. Then $c^* = a'' \cup d' \in \mathcal{P}(a^\alpha, d^\delta)$. Moreover, c^* and c represent adjacent vertices in $\mathcal{A}(S)$, as desired. □

Note that we did not care whether c was in minimal position with d or not. A slight enhancement shows that the triangles are 1–centred:

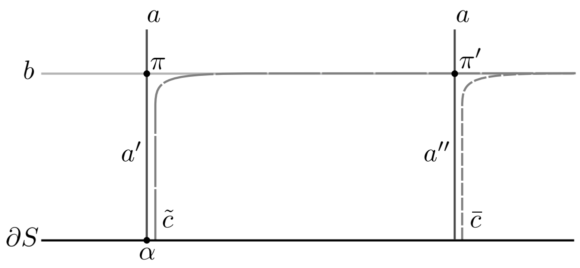

FIGURE 1. The only possible half-bigon between $\tilde{c}$ and a

Lemma 3.4. *Suppose that we have arcs with endpoints $a^\alpha, b^\beta, d^\delta$, mutually in minimal position. Then there are pairwise adjacent vertices on $\mathcal{P}(a^\alpha, b^\beta), \mathcal{P}(a^\alpha, d^\delta)$ and $\mathcal{P}(d^\delta, b^\beta)$.*

Proof. If two of a, b, d are disjoint, then there is nothing to prove. Otherwise for unicorn arcs $c_i = a' \cup b', c_{i+1} = a'' \cup b''$ let π, σ their intersection points with d closest to δ along d. There is $0 \leq i < n$ such that $\pi \in a', \sigma \in b''$. Without loss of generality assume that π is not farther than σ from δ. Let π' be the intersection point of a with the subarc $\delta\sigma \subset d$ that is closest to α along a. Then c_{i+1}, the unicorn arc obtained from d^δ, b^β determined by σ, and the unicorn arc obtained from a^α, d^δ determined by π', represent three adjacent vertices in $\mathcal{A}(S)$. $\square$

We now prove that unicorn paths are invariant under taking subpaths, up to one exception.

Lemma 3.5. *For every $0 \leq i < j \leq n$, either $\mathcal{P}(c_i^\alpha, c_j^\beta)$ is a subpath of $\mathcal{P}(a^\alpha, b^\beta)$, or $j = i+2$ and c_i, c_j represent adjacent vertices of $\mathcal{A}(S)$.*

Before we give the proof, we need the following.

Sublemma 3.6. *Let $c = c_{n-1}$, which means that $c = a' \cup b'$ with the interior of a' disjoint from b. Let $\tilde{c}$ be the arc homotopic to c obtained by homotopying a' slightly off a so that $a' \cap \tilde{c} = \emptyset$. Then either $\tilde{c}$ and a are in minimal position, or they bound exactly one half-bigon, shown in Figure 1. In that case, after homotopying $\tilde{c}$ through that half-bigon to $\bar{c}$, the arcs $\bar{c}$ and a are already in minimal position.*

Proof. Let $\tilde{\alpha}$ be the endpoint of $\tilde{c}$ corresponding to α in c. The arcs $\tilde{c}$ and a cannot bound a bigon, since then b and a would bound a bigon contradicting minimal position. Hence if $\tilde{c}$ and a are not in minimal position, then they bound a half-bigon $\tilde{c}'a''$, where $\tilde{c}' \subset \tilde{c}, a'' \subset a$. Let $\pi' = \tilde{c}' \cap a''$. The subarc $\tilde{c}'$ contains $\tilde{\alpha}$, since otherwise a and b would bound a half-bigon. Since the interior of a' is disjoint from b, by minimal position of a and b the interior of a'' is also disjoint from b. In particular, a'' does not contain α, since otherwise $a' \subsetneq a''$ and π would lie in the interior of a''. Moreover, π and π' are consecutive intersection points with a on b (see Figure 1).

Let b'' be the component of $b - \pi'$ containing β. Let $\bar{c}$ be obtained from $a'' \cup b''$ by homotopying it off a''. Applying to $\bar{c}$ the same argument as to $\tilde{c}$, but with the endpoints of a interchanged, we get that either $\bar{c}$ is in minimal position with a or there is a a half-bigon $\bar{c}'a'''$, where $\bar{c}' \subset \bar{c}, a''' \subset a$. But in the latter case we have $\alpha \in a'''$, which implies $a' \subsetneq a'''$ contradicting the fact that the interior of a''' should be disjoint from b. $\square$

Proof of Lemma 3.5. We can assume $i = 0$, so that $c_i = a$, and $j = n-1$, so that $c_j = a' \cup b'$, where a' intersects b only at its endpoint π distinct from α. Let $\tilde{c}$ be

obtained from $c = c_j$ as in Sublemma 3.6. If $\tilde{c}$ is in minimal position with a, then points in $(a \cap b) - \pi$ determining unicorn arcs obtained from a^α, b^β determine the same unicorn arcs obtained from $a^\alpha, \tilde{c}^\beta$, and exhaust them all, so we are done.

Otherwise, let $\bar{c}$ be the arc from Sublemma 3.6 homotopic to c and in minimal position with a. The points $(a \cap b) - \pi - \pi'$ determining unicorn arcs obtained from a^α, b^β determine the same unicorn arcs obtained from $a^\alpha, \bar{c}^\beta$. Let $a^* = a - a''$. If π' does not determine a unicorn arc obtained from a^α, b^β, i.e. if a^* and b'' intersect outside π', then we are done as in the previous case. Otherwise, $a^* \cup b'' = c_1$, since it is minimal in the order on the unicorn arcs obtained from a^α, b^β. Moreover, since the subarc $\pi\pi'$ of a lies in a^*, its interior is disjoint from b'', hence also from b'. Thus $a^* \cup b''$ precedes c in the order on the unicorn arcs obtained from a^α, b^β, which means that $j = 2$, as desired. □

4. Arc graphs are hyperbolic

Definition 4.1. To a pair of vertices a, b of $\mathcal{A}(S)$ we assign the following family $P(a, b)$ of unicorn paths. Slightly abusing the notation we realise them as arcs a, b on S in minimal position. If a, b are disjoint, then let $P(a, b)$ consist of a single path (a, b). Otherwise, let α_+, α_- be the endpoints of a and let β_+, β_- be the endpoints of b. Define $P(a, b)$ as the set of four unicorn paths: $\mathcal{P}(a^{\alpha_+}, b^{\beta_+}), \mathcal{P}(a^{\alpha_+}, b^{\beta_-}), \mathcal{P}(a^{\alpha_-}, b^{\beta_+})$, and $\mathcal{P}(a^{\alpha_-}, b^{\beta_-})$.

The proof of the next proposition follows along the lines of [Ham07, Prop 3.5] (or [BH99, Thm III.H.1.7]).

Proposition 4.2. *Let $\mathcal{G}$ be a geodesic in $\mathcal{A}(S)$ between vertices a, b. Then any vertex $c \in \mathcal{P} \in P(a, b)$ is at distance ≤ 6 from $\mathcal{G}$.*

In the proof we need the following lemma which is immediately obtained by applying k times Lemma 3.3.

Lemma 4.3. *Let $x_0, \ldots, x_m$ with $m \leq 2^k$ be a sequence of vertices in $\mathcal{A}(S)$. Then for any $c \in \mathcal{P} \in P(x_0, x_m)$ there is $0 \leq i < m$ with $c^* \in \mathcal{P}^* \in P(x_i, x_{i+1})$ at distance $\leq k$ from c.*

Proof of Proposition 4.2. Let $c \in \mathcal{P} \in P(a, b)$ be at maximal distance k from $\mathcal{G}$. Assume $k \geq 1$. Consider the maximal subpath $a'b' \subset \mathcal{P}$ containing c with a', b' at distance $\leq 2k$ from c. By Lemma 3.5 we have $a'b' \in P(a, b)$. Let $a'', b'' \in \mathcal{G}$ be closest to a', b'. Thus $|a'', a'| \leq k, |b'', b'| \leq k$, and in the case where $a' = a$ or $b' = b$, we have $a'' = a$ or $b'' = b$ as well. Hence $|a'', b''| \leq 6k$. Consider the concatenation of $a''b''$ with any geodesic paths $a'a'', b''b'$. Denote the consecutive vertices of that concatenation by $x_0, \ldots, x_m$, where $m \leq 8k$. By Lemma 4.3, the vertex c is at distance $\leq \lceil \log_2 8k \rceil$ from some x_i. If $x_i \notin \mathcal{G}$, say $x_i \in a'a''$ then $|c, x_i| \geq |c, a'| - |a', x_i| \geq k$, so that $\lceil \log_2 8k \rceil \geq k$. Otherwise if $x_i \in \mathcal{G}$, then we also have $\lceil \log_2 8k \rceil \geq k$, this time by the definition of k. This gives $k \leq 6$. □

Proof of Theorem 1.2. Let abd be a triangle in $\mathcal{A}(S)$ formed by geodesic edge-paths. By Lemma 3.4, there are pairwise adjacent vertices c_{ab}, c_{ad}, c_{db} on some paths in $P(a, b), P(a, d), P(b, d)$. We now apply Proposition 4.2 to c_{ab}, c_{ad}, c_{db} finding vertices on ab, ad, bd at distance ≤ 6. Thus abd is 7–centred at c_{ab}. □

5. Curve graphs are hyperbolic

In this section let $|\cdot, \cdot|$ denote the combinatorial distance in $\mathcal{AC}(S)$ instead of in $\mathcal{A}(S)$.

Remark 5.1 ([MM00, Lem 2.2]). Suppose that $\mathcal{C}(S)$ is connected and hence S is not the four holed sphere or the once holed torus. Consider a retraction $r\colon \mathcal{AC}^{(0)}(S) \to \mathcal{C}^{(0)}(S)$ assigning to each arc a boundary component of a regular neighbourhood of its union with ∂S. We claim that r is 2–Lipschitz. If S is not the twice holed torus, the claim follows from the fact that a pair of disjoint arcs does not fill S. Otherwise, assume that a, b are disjoint arcs filling the twice holed torus S. Then the endpoints of a, b are all on the same component of ∂S and $r(a), r(b)$ is a pair of curves intersecting once. Hence the complement of $r(a)$ and $r(b)$ is a twice holed disc, so that $r(a), r(b)$ are at distance 2 in $\mathcal{C}(S)$ and the claim follows.

Moreover, if b is a curve in $\mathcal{AC}^{(0)}(S)$ adjacent to an arc a, then b is adjacent to $r(a)$ as well. Thus the distance in $\mathcal{C}(S)$ between two nonadjacent vertices c, c' does not exceed $2|c, c'| - 2$. Consequently, a geodesic in $\mathcal{C}(S)$ is a 2–quasigeodesic in $\mathcal{AC}(S)$. Here we say that an edge-path with vertices $(c_i)_i$ is a 2*–quasigeodesic*, if $|i - j| \leq 2|c_i, c_j|$.

Proof of Theorem 1.1. We first assume that S has nonempty boundary. Let $T = abd$ be a triangle in the curve graph formed by geodesic edge-paths. By Remark 5.1, the sides of T are 2–quasigeodesics in $\mathcal{AC}(S)$. Choose arcs $\bar{a}, \bar{b}, \bar{d} \in \mathcal{AC}^{(0)}(S)$ that are adjacent to a, b, d, respectively.

Let k be the maximal distance from any vertex $\bar{c} \in \mathcal{P} \in P(\bar{a}\bar{b})$ to the side $\mathcal{G} = ab$. Assume $k \geq 1$. As in the proof of Proposition 4.2, consider the maximal subpath $a'b' \subset \mathcal{P}$ containing $\bar{c}$ with a', b' at distance $\leq 2k$ from $\bar{c}$. Let $a'', b'' \in \mathcal{G}$ be closest to a', b', so that $|a'', b''| \leq 6k$. Consider the concatenation $(x_i)_{i=0}^m$ of $a''b''$ with any geodesic paths $a'a'', b''b'$ in $\mathcal{AC}(S)$. Since $a''b''$ is a 2–quasigeodesic, we have $m \leq 2k + 2|a'', b''| = 14k$. For $i = 0, \ldots, m-1$ let $\bar{x}_i \in \mathcal{AC}^{(0)}(S)$ be an arc adjacent (or equal) to both x_i and x_{i+1}. Note that then all paths in $P(\bar{x}_i, \bar{x}_{i+1})$ are at distance 1 from x_{i+1}. By Lemmas 3.5 and 4.3, the vertex $\bar{c}$ at distance $\leq \lceil \log_2 14k \rceil$ from a path in some $P(\bar{x}_i, \bar{x}_{i+1})$. Hence $\lceil \log_2 14k \rceil + 1 \geq k$. This gives $k \leq 8$.

By Lemma 3.4, there are pairwise adjacent vertices on some paths in $P(\bar{a}, \bar{b}), P(\bar{a}, \bar{d})$, and in $P(\bar{b}, \bar{d})$. Let $\bar{c}$ be one of these vertices. Then $\bar{c}$ is at distance ≤ 9 from all the sides of T in $\mathcal{AC}(S)$. Consider the curve $c = r(\bar{c})$ adjacent to $\bar{c}$, where r is the retraction from Remark 5.1. Then T considered as a triangle in $\mathcal{C}(S)$ is 17–centred at c, by Remark 5.1. Hence $\mathcal{C}(S)$ is 17–hyperbolic for $\partial S \neq \emptyset$.

The curve graph $\mathcal{C}(S)$ of a closed surface (if connected) is known to be a 1–Lipschitz retract of the curve graph $\mathcal{C}(S')$, where S' is the once punctured S [Har86, Lem 3.6], [RS11, Thm 1.2]. The retraction is the puncture forgetting map. A section $\mathcal{C}(S) \to \mathcal{C}(S')$ can be constructed by choosing a hyperbolic metric on S, realising curves as geodesics and then adding a puncture outside the union of the curves. Hence $\mathcal{C}(S)$ is 17–hyperbolic as well. □

References

[ABC+91] J. M. Alonso, T. Brady, D. Cooper, V. Ferlini, M. Lustig, M. Mihalik, M. Shapiro, and H. Short, *Notes on word hyperbolic groups*, Group theory from a geometrical viewpoint (Trieste, 1990), World Sci. Publ., River Edge, NJ, 1991, pp. 3–63. Edited by Short.

[Aou12] Tarik Aougab, *Uniform Hyperbolicity of the Graphs of Curves* (2012), available at `arXiv:1212.3160`.

[Bow06] Brian H. Bowditch, *Intersection numbers and the hyperbolicity of the curve complex*, J. Reine Angew. Math. **598** (2006), 105–129.

[Bow12] ———, *Uniform hyperbolicity of the curve graphs* (2012), available at `http://homepages.warwick.ac.uk/~masgak/papers/uniformhyp.pdf`.

[BH99] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999.

[CRS13] M.T. Clay, K. Rafi, and S. Schleimer, *Uniform hyperbolicity of the curve graph via surgery sequences* (2013), in preparation.

[Ham07] Ursula Hamenstädt, *Geometry of the complex of curves and of Teichmüller space*, Handbook of Teichmüller theory. Vol. I, IRMA Lect. Math. Theor. Phys., vol. 11, Eur. Math. Soc., Zürich, 2007, pp. 447–467.

[Har86] John L. Harer, *The virtual cohomological dimension of the mapping class group of an orientable surface*, Invent. Math. **84** (1986), no. 1, 157–176.

[Hat91] Allen Hatcher, *On triangulations of surfaces*, Topology Appl. **40** (1991), no. 2, 189–194.

[HOP12] Sebastian Hensel, Damian Osajda, and Piotr Przytycki, *Realisation and dismantlability* (2012), available at `arXiv:1205.0513`.

[HH12] Arnaud Hilion and Camille Horbez, *The hyperbolicity of the sphere complex via surgery paths* (2012), available at `arXiv:1210.6183`.

[MM99] Howard A. Masur and Yair N. Minsky, *Geometry of the complex of curves. I. Hyperbolicity*, Invent. Math. **138** (1999), no. 1, 103–149.

[MM00] H. A. Masur and Y. N. Minsky, *Geometry of the complex of curves. II. Hierarchical structure*, Geom. Funct. Anal. **10** (2000), no. 4, 902–974.

[MS13] Howard Masur and Saul Schleimer, *The geometry of the disk complex*, J. Amer. Math. Soc. **26** (2013), no. 1, 1–62.

[RS11] Kasra Rafi and Saul Schleimer, *Curve complexes are rigid*, Duke Math. J. **158** (2011), no. 2, 225–246.

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